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Surmounting fluctuating barriers: A simple model in discrete time

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The escape rate is calculated for a piecewise linear map in the presence of two additive weak noises: Gaussian white noise (thermal noise) and Ornstein-Uhlenbeck noise (barrier fluctuations). A transition state theory yields the exact escape rate in the weak noise limit. By including the dominant finite-noise corrections we find very good agreement with numerical simulations. Of particular interest is the behavior of the escape rate $k(\tau)$ as a function of the correlation time τ of the Ornstein-Uhlenbeck noise. We show that the qualitative behavior of $k(\tau)$ at any fixed thermal noise intensity is the same as in the absence of thermal noise. Whether $k(\tau)$ exhibits a local maximum (resonant activation) is determined by the detailed τ dependence of the colored-noise intensity: e.g., resonant activation is found (at a correlation time of order 1) if the integral autocorrelation of the colored noise is kept τ independent but not if the variance is kept τ independent. In the continuous-time limit neither of these cases shows resonant activation.

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I. INTRODUCTION

Nonlinear dynamical systems with fluctuating parameters in a thermal environment have been investigated in various physical contexts [1–3]. Only very recently, Doering and Gadoua [4] detected that the escape rate over a fluctuating barrier may exhibit a maximum resonant activation as a function of the correlation time of the barrier fluctuations. They considered an overdamped Brownian particle driven by Gaussian white noise in a piecewise linear double-well potential whose slope switches between two values according to a dichotomous process. Their limitation to extremely large barrier fluctuations could be relaxed in subsequent investigations of the same model [5,6], showing that resonant activation occurs for arbitrary weak thermal noise and barrier fluctuations at correlation times comparable to the inverse escape rate. Some light on the basic mechanism leading to the astonishing phenomenon [7] of resonant activation was shed by the study [8] of even much simpler models which, under appropriate conditions, exhibit the same behavior.

In this paper we address the question of whether resonant activation can also be found for other kinds of potentials and barrier fluctuations than in the Doering-Gadoua model [4–6]. We study a discrete-time dynamics additively coupled to thermal fluctuations in the form of weak Gaussian white noise. In order to arrive at rigorous results, the deterministic part of the dynamics is described by an extremely simple map consisting of two linear pieces, and the barrier fluctuations are modeled by weak additive Ornstein-Uhlenbeck noise. We also discuss the continuous-time limit of our model corresponding to a piecewise parabolic potential with a cusp. In continuous time the same kind of system has recently been investigated [9] within the unified colored-noise approximation for general potentials and multiplicative couplings of the Ornstein-Uhlenbeck noise.

We proceed as follows: In Sec. II our discrete-time model is introduced. Section III starts with a brief summary of the calculation of the escape rate which is elaborated in detail in the Appendix by means of a transition state theory. The central rate formula is given in Eqs. (8)–(10), the discussion of which is carried out in the remainder of Sec. III. Section IV deals with the continuous-time limit of our model. Finally, Sec. V gives a summary and our conclusions.

II. THE MODEL

We consider the following two-dimensional Markov process in discrete time n:

$$x_{n+1}=f(x_n,y_n)+\sqrt{K_1}\xi_n^{(1)} , \tag{1}$$

$$y_{n+1}=Ay_n+\sqrt{K_2}\xi_n^{(2)} , \tag{2}$$

where $\xi_n^{(i)}$, $i=1,2$, is identically distributed Gaussian white noise,

$$P(\xi_n^{(i)})=\frac{1}{\sqrt{\pi\epsilon}}e^{-\xi_n^{(i)2}/\epsilon}, \quad \langle \xi_n^{(i)}\xi_m^{(j)}\rangle=\frac{\epsilon}{2}\delta_{ij}\delta_{nm} , \tag{3}$$

of small noise strength ϵ. In (1), x_n represents a particle under the influence of weak thermal noise $\xi_n^{(1)}$ in a fluctuating environment described by y_n. The non-negative coupling strengths K_i of the white noise are typically of order 1 but the limiting cases that one of them vanishes are also of certain interest.

As for y_n, it follows from (2) and (3) that

$$\langle y_{n+l+1}y_n\rangle=A\langle y_{n+l}y_n\rangle$$

for $l\geq 0$ and, thus,

$$\langle y_{n+l}y_n\rangle=A^l\langle y_n^2\rangle . \tag{4}$$

We restrict ourselves to the case that the y process described by (2) is in the stationary state, which is possible only for

$$|A|<1 . \quad (5)$$

Then (2) and (4) imply that y_n are Gaussian random numbers of vanishing mean and exponentially decaying correlation, i.e., they represent Ornstein-Uhlenbeck noise. For the sake of brevity we will call A the correlation in contrast to the correlation time, which is given by $[\ln|A|^{-1}]^{-1}$.

As mentioned in the Introduction, in order to arrive at rigorous results, the map $f(x,y)$ in (1) is assumed to be of the very simple form

$$f(x,y)=\begin{cases} Bx+y & \text{for } x\le 1 \\ 2+B(x-2)+y & \text{for } x>1 . \end{cases} \quad (6)$$

With the restriction

$$|B|<1 , \quad (7)$$

it follows that in the absence of any noise, i.e., $K_1=0$ and $y_n=0$, the deterministic dynamics (1) has two point attractors at $x=0$ and 2. In the presence of weak noise a particle still spends most of its time in a close neighborhood of $x=0$ or 2, but once in a while it switches between these domains. The probability per time step to switch is given by the escape rate k which is the central quantity throughout the following investigation. Note that without thermal noise, $K_1=0$, our model reduces to the discrete-time version [10] of the common escape problem for a particle that is additively disturbed by weak Ornstein-Uhlenbeck noise [11].

III. DISCUSSION OF THE ESCAPE RATE

The calculation of the escape rate is carried out in the Appendix by means of a transition state theory. Here we only summarize the main steps. First, the invariant density for the fully linear system (1)–(3), (A1) is calculated, resulting in a Gaussian distribution centered at the origin. Then it is shown that the pseudoinvariant density describing the escapes in the piecewise linear model (6) agrees with the invariant density of the fully linear system (A1) in those regions which mainly count for the rate. Finally, it is pointed out that only single crossings of the basin boundary at $x=1$ have to be taken into account for the rate, whereas recrossings are negligible due to the discontinuity of the map (6) at $x=1$. The final rate formula is given by

$$k=\left[\frac{\epsilon}{4\pi\Delta\phi}\right]^{1/2}\exp\left\{-\frac{\Delta\phi}{\epsilon}\right\}R\left[\frac{\epsilon}{\Delta\phi}\right] , \quad (8)$$

$$\Delta\phi=\frac{(1-A^2)(1-AB)(1-B^2)}{(1-A^2)(1-AB)K_1+(1+AB)K_2} , \quad (9)$$

$$\begin{aligned} R\left[\frac{\epsilon}{\Delta\phi}\right] &= \int_0^\infty dx \exp\left\{-x\left[1+\frac{x}{4}\frac{\epsilon}{\Delta\phi}\right]\right\} \\ &=1-\frac{1}{2}\frac{\epsilon}{\Delta\phi}+\frac{1}{2}\frac{3}{2}\left[\frac{\epsilon}{\Delta\phi}\right]^2 \\ &\quad -\frac{1}{2}\frac{3}{2}\frac{5}{2}\left[\frac{\epsilon}{\Delta\phi}\right]^3+\cdots . \end{aligned} \quad (10)$$

The dominating weak noise behavior of the rate, i.e., (8) without $R(\epsilon/\Delta\phi)$, is of the same structure as was derived in [12] for general maps with point attractors which are disturbed by weak Gaussian *white* noise: it consists of an Arrhenius factor and a preexponential factor proportional to $\sqrt{\epsilon}$. The complete right-hand side (rhs) of (8) represents the exact rate up to exponentially small corrections in ϵ, i.e., up to a factor of the form $1+O(e^{-c/\epsilon})$ with an ϵ-independent $c>0$. Thus, $R(\epsilon/\Delta\phi)$ accounts for all nonexponential finite-ϵ corrections of the dominating weak noise behavior. Note that the series expansion of $R(\epsilon/\Delta\phi)$ in (10) is semiconvergent. Interestingly enough, the rate (8) including the definition (10) of $R(\epsilon/\Delta\phi)$ is exactly identical to the result for a *continuous* piecewise linear map in the presence of additive Gaussian *white* noise [13] (of course, with a different $\Delta\phi$). Clearly, the rate (8) is invariant under $A\mapsto -A$ and $B\mapsto -B$, which can be easily understood in terms of most probable escape paths as introduced in the Appendix. The comparison of the rate formula (8) with numerical simulations is given in Table I.

Next we consider the behavior of the rate as a function of the correlation A. Apart from the finite-ϵ corrections $R(\epsilon/\Delta\phi)$, the inverse rate (8) shows the same qualitative behavior (e.g., extrema) as $\Delta\phi$. In particular, resonant activation corresponds to a minimum of $\Delta\phi$. It thus suffices to study $\Delta\phi$ as a function of A. We furthermore take over the usual assumption [4–6,9] that the coupling strength K_1 of the thermal noise is independent of the correlation A of the barrier fluctuations, whereas the coupling strength K_2 of the Ornstein-Uhlenbeck noise may be a nontrivial function of the correlation, $K_2=K_2(A)$.

The quantity $\Delta\phi$ in (9) can be rewritten as

$$\begin{aligned} \Delta\phi &= \left[\left[\frac{1-B^2}{K_1}\right]^{-1} + \left[\frac{(1-A^2)(1-AB)(1-B^2)}{(1+AB)K_2}\right]^{-1}\right]^{-1} \\ &=[(\Delta\phi_{K_2=0})^{-1}+(\Delta\phi_{K_1=0})^{-1}]^{-1} . \end{aligned} \quad (11)$$

Since $\Delta\phi_{K_2=0}$ is A independent, $\Delta\phi$ as a function of A has an extremum if and only if $\Delta\phi_{K_1=0}$ has an extremum. An extremum of $\Delta\phi$ is most pronounced for $K_1=0$ and flattens with increasing K_1. In other words, every kind of resonant activation is already present without thermal noise, $K_1=0$, and gets even weaker in the presence of the thermal noise. Thus, in our model resonant activation is not the effect of an interplay between thermal and barrier fluctuations but rather a property of the rate for the escape problem driven by colored noise only, which survives in the presence of additional white noise.

It is clear that by an appropriate choice of $K_2(A)$ arbitrary "resonant" extrema of $\Delta\phi$ in (9) can be induced. For example, from (11) one easily sees that a local minimum of $\Delta\phi$ and thus resonant activation will be found whenever $K_2(A)$ disappears faster than proportional to $1-A^2$ for $A\to\pm 1$. On the other hand, if $K_2(A)$ stays larger than proportional to $1-A^2$ for

$A \to \pm 1$, then the opposite of resonant activation, namely, a local maximum of $\Delta\phi$ will arise. The appearance of additional local extrema of $\Delta\phi$ in the above cases and the behavior of $\Delta\phi$ for other asymptotics of $K_2(A)$ for $A \to \pm 1$ depend on the details of $K_2(A)$ and possibly also on the particular value of B.

Unfortunately, our model is too far from a realistic physical situation to suggest a certain kind of function $K_2(A)$. Instead we will discuss in more detail three different choices for $K_2(A)$ which are of interest for themselves and are often considered in the literature. The discussion is given in terms of $\Delta\phi$, but, as seen above, the same properties carry over to $\Delta\phi_{K_1=0}$ describing the escape problem with Ornstein-Uhlenbeck noise alone.

We first address the case that K_2 is A independent. Then $\Delta\phi$ in (9) as a function of $A \in (-1,1)$ has a maximum at $-1 < A_{\max} < 0$ for $B > 0$ and at $0 < A_{\max} < 1$ for $B < 0$, is strictly monotonous on both sides of $A_{\max}$, and vanishes for $A \to \pm 1$. In agreement with the above general discussion, we thus find the opposite of resonant activation, namely, a minimum of the escape rate (8) as a function of the correlation A (see Fig. 1).

Next we choose $K_2(A)$ such that the variance $\langle y^2 \rangle$ of the barrier fluctuations is A independent in analogy to the Doering-Gadoua model [4–6]. From the stationary distribution (A15) of the barrier fluctuations derived in the Appendix, we find that

$$\langle y^2 \rangle = \frac{\epsilon K_2(A)}{2(1-A^2)} . \tag{12}$$

Thus, we have

$$K_2(A) = (1-A^2) K_2(0) .$$

It follows that $\Delta\phi$ in (9) is strictly monotonously decreasing or increasing for $B > 0$ or $B < 0$, respectively, and stays finite for $A \to \pm 1$. In other words, normalizing the barrier fluctuations in analogy to the Doering-Gadoua model does not lead to resonant activation in our model (see Fig. 1).

We finally consider the case that the integral autocorrelation $\sum_{n=0}^{\infty} \langle y_0 y_n \rangle$ of the barrier fluctuations is

TABLE I. Comparison of the theoretical escape rate $k_{\rm th}$ (8)–(10) with results $k_{\rm num}$ from numerical simulations of the Langevin equation (1)–(3) and (6) for different values of the correlation A and the slope B. The coupling strengths are chosen $K_1 = 1$ and $K_2 = 1 - A^2$. The latter choice yields a constant variance of the barrier fluctuations, [see text below (12)], and a $\Delta\phi$ in (9) that stays finite but different from the case $K_2 = 0$ for $A \to \pm 1$ and is invariant under $A \mapsto -A$, $B \mapsto -B$. The Table shows the relative difference in percent $100(k_{\rm th} - k_{\rm num})/k_{\rm th}$ between the numerical and the theoretical rates for three different values of the noise strength ϵ. For any value of A and B the noise strength was chosen such that the quantity $\Delta\phi/\epsilon$ equals 2, 5, or 8. Thus the theoretical rates $k_{\rm th}$ in (8) are equal to $2.275\ldots \times 10^{-2}$, $7.827\ldots \times 10^{-4}$, or $3.167\ldots \times 10^{-5}$, respectively. In order that the initial points (x_0, y_0) in the Langevin equation (1) and (2) be distributed according to the pseudoinvariant density, every realization was started at the origin, followed by 100 preliminary iterations. The resulting position was taken as initial point (x_0, y_0) if $x_0 < 1$ and rejected else. Then the escape time n was measured until x_n arrived in the neighborhood [1.5,2.5] of the point attractor at $x = 2$ for the first time. The rate $k_{\rm num}$ was calculated as the inverse mean escape time from 10^4 realizations, implying a statistical uncertainty of 1%. Even for the rather modest values of $\Delta\phi/\epsilon$ considered here, the agreement between the numerical and theoretical rates is very good except for B close to ± 1. The latter discrepancies can be attributed to recrossings of the basin boundary at $x = 1$ which are neglected in the theoretical rate. Also the prediction that the relative differences between the numerical and theoretical rates are of order $e^{-c/\epsilon}$ with an ϵ-independent (but A and B dependent) $c > 0$ turns out to be satisfied rather well.

B \ A	$\Delta\phi/\epsilon$	-0.9	-0.6	-0.3	0.0	0.3	0.6	0.9
-0.9	2	77	66	64	63	63	62	63
-0.9	5	68	47	42	41	40	41	42
-0.9	8	60	34	28	26	26	26	25
-0.6	2	50	23	16	17	17	15	29
-0.6	5	30	9	4	3	1	5	5
-0.6	8	16	2	0	0	0	0	0
-0.3	2	34	10	5	2	6	9	32
-0.3	5	15	3	1	-1	0	1	6
-0.3	8	6	8	1	0	2	-1	1
0.0	2	9	12	5	5	6	15	43
0.0	5	8	0	1	1	1	3	12
0.0	8	1	-1	0	0	1	1	3
0.3	2	29	21	18	18	20	29	57
0.3	5	16	8	4	1	4	9	27
0.3	8	10	3	2	-1	1	3	11
0.6	2	44	42	41	41	41	49	70
0.6	5	28	22	16	16	17	28	51
0.6	8	23	16	7	5	10	16	35
0.9	2	80	80	80	80	81	82	89
0.9	5	64	64	63	62	63	68	82
0.9	8	52	51	49	47	49	57	76

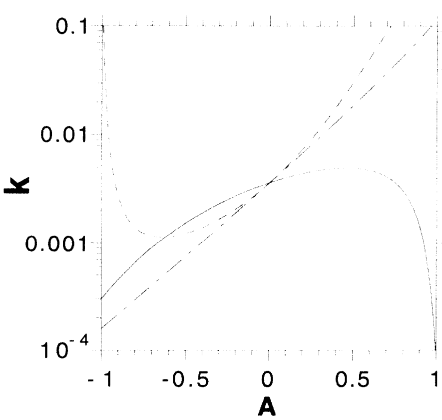

FIG. 1. The escape rate k in (8) as a function of the correlation A for $B = 0.8$, $K_1 = 1$, and $\epsilon = 0.05$ with fixed coupling strength $K_2 = 1$ (dashed curve), fixed variance $K_2 = (1 - A^2)$ (dot-dashed curve), and fixed integral autocorrelation $K_2 = (1 - A^2)(1 - A)$ (solid curve) of the barrier fluctuations.

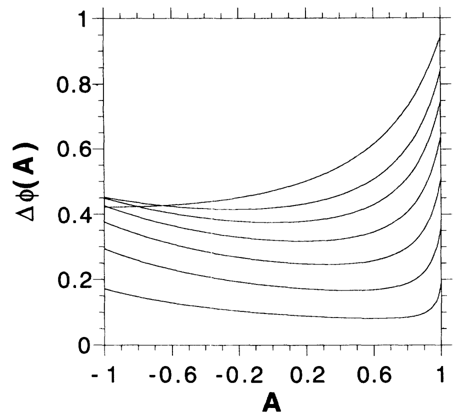

FIG. 2. $\Delta\phi$ in (9) as a function of the correlation A for $B=\sqrt{5}-2=0.236\ldots$, $0.4, 0.5, \ldots, 0.9$ (from above). The coupling strengths in (9) are chosen $K_1=1$ and $K_2=(1-A^2)(1-A)$; cf. (13). Note that a moderate minimum in $\Delta\phi(A)$ yields a very pronounced maximum (resonant activation) of the rate (8) for small noise strengths ϵ (see also Fig. 1).

kept A independent. This choice is common in studies of continuous-time systems with colored noise because it gives a sensible white noise limit [11]. With (4) and (12) we arrive at

$$K_2(A)=(1-A^2)(1-A)K_2(0)\ . \tag{13}$$

Note that with (13), $\Delta\phi$ in (9) is no longer invariant under $A\mapsto -A$ and $B\mapsto -B$. In the limit $A\to -1$, the quantity $\Delta\phi$ stays finite. For $A\to +1$, we find that $\Delta\phi=\Delta\phi_{K_2=0}$, i.e., the barrier fluctuations disappear in this limit. Clearly, $\Delta\phi=\Delta\phi_{K_2=0}$ stays finite provided $K_1>0$. Moreover, $\Delta\phi$ has a local minimum $A_{\min}$ within the allowed domain $(-1,1)$ of correlations (5) for

$$\sqrt{5}-2=0.236\ldots<B<1$$

given by

$$A_{\min}=\frac{1-\sqrt{2(1-B)}}{B}\ . \tag{14}$$

The local minimum is most pronounced, i.e., $\Delta\phi(A=\pm 1)/\Delta\phi(A_{\min})$ takes the largest values for $B\to 1$, see Fig. 2. Thus, for fixed integral autocorrelation of the barrier fluctuations, resonant activation can be found in our model (see Fig. 1). Except for B very close to 1 or $\sqrt{5}-2$, the resonant correlation time $[\ln|A_{\min}|^{-1}]^{-1}$, [see text below (5)] is of order 1.

IV. CONTINUOUS-TIME LIMIT

Let us denote by Δt the time step in the discrete-time model (1)–(3) and (6). Then the continuous-time limit is found [10,14] by introducing $x(t=n\Delta t):=x_n$, $y(t=n\Delta t):=y_n/\Delta t$, $b:=(1-B)/\Delta t$, $\tau:=\Delta t/(1-A)$, $D_1:=\epsilon K_1/[4\Delta t]$, and

$$D_2:=\epsilon K_2/[4\Delta t(1-A)^2]$$

and then letting Δt go to zero. One recovers the common Langevin equations [9,11]

$$\dot{x}(t)=-U'[x(t)]+y(t)+\sqrt{2D_1}\,\xi_1(t)\ , \tag{15}$$

$$\dot{y}(t)=-\frac{1}{\tau}y(t)+\frac{\sqrt{2D_2}}{\tau}\xi_2(t)\ , \tag{16}$$

where $\xi_i(t)$, $i=1,2$, is Gaussian noise of vanishing mean and correlation

$$\langle \xi_i(t)\xi_j(s)\rangle=\delta_{ij}\delta(t-s)\ . \tag{17}$$

The correlation time of the Ornstein-Uhlenbeck noise $y(t)$ is given by τ, whereas the quantities D_1 and D_2 characterize the strengths of the white noise $\xi_1(t)$ and the colored noise $y(t)$ in (15), respectively. In our model, the potential $U(x)$ is piecewise parabolic with a cusp at $x=1$:

$$U(x)=\begin{cases}\dfrac{b}{2}x^2 & \text{for } x\le 1\\[2mm] \dfrac{b}{2}(x-2)^2 & \text{for } x\ge 1\ .\end{cases} \tag{18}$$

Concerning the great practical importance of cusp-shaped potentials, we refer to Sec. 7.E.2 of the review [15]. The additive Ornstein-Uhlenbeck noise $y(t)$ in (15) leads to an asymmetric fluctuation of the potential depths and the location of the minima but not of the curvatures nor of the position of the cusp.

In the continuous-time limit, the results found in the Appendix yield the invariant density $W(x,y)$ of the dynamics (15)–(17) with a fully parabolic potential $U(x)=(b/2)x^2$:

$$W(x,y)=N\exp\left\{-\frac{C}{2[D_1C+D_2]}\left[bx^2+\tau\frac{\left[bx-\left[1+\frac{D_1}{D_2}C\right]y\right]^2}{1+\frac{D_1}{D_2}C^2}\right]\right\}, \tag{19}$$

where $C:=1+b\tau$ and N is a normalization constant. The validity of this result is readily checked by insertion into the Fokker-Planck equation corresponding to (15) and (16). Then the usual transition state theory in continuous time [15] leads to the exponentially leading part of the escape rate k for the piecewise parabolic potential (18):

$$-\ln k=\frac{b}{2}\frac{1+b\tau}{(1+b\tau)D_1+D_2} \tag{20}$$

for small noise strengths D_i. This result agrees with the exponentially leading part of the discrete-time rate (8) in the continuous-time limit. The determination of the preexponential part of the rate for a cusp-shaped barrier in the presence of colored noise is a well-known unsolved problem. In particular, the preexponential part of the rate (8) becomes wrong in the continuous-time limit, as can be seen by comparison with the known result for the special case of white noise $D_2=0$ (see Sec. 7.E.2 in [15]).

The discussion of the rate (20) is similar to the discrete-time case: Assuming that the white noise strength D_1 is independent of the correlation time τ, the rhs of (20) can be rewritten analogous to (11), showing that the qualitative behavior is the same for any choice of D_1, in particular $D_1=0$. One easily sees that both a τ-independent variance $\langle y(t)^2\rangle$ of the barrier fluctuations [implying $D_2(\tau)=D_2(0)/\tau$] as well as a τ-independent integral autocorrelation $\int_0^\infty \langle y(t)y(0)\rangle dt$ [implying $D_2(\tau)=D_2(0)=\text{const}$] do *not* lead to resonant activation, at least in the exponentially leading part of the rate considered here.

Recently, the escape problem (15)–(17) has been solved [9] by means of a generalized unified colored-noise approximation (UCNA) for very general potentials $U(x)$ and multiplicative coupling of the Ornstein-Uhlenbeck noise, i.e., with an additional function $g[x(t)]$ multiplying $y(t)$ in (15). For our special model (15)–(18) the exponentially leading part of the UCNA result is given by

$$-\ln k = \frac{b}{2}\frac{1+b\tau}{D_1+D_2} \tag{21}$$

for small noise strengths D_i. This expression agrees with the exact result (20) in the absence of thermal noise, $D_1=0$, and for vanishing correlation time $\tau=0$ of the barrier fluctuations, but not in general (a similar observation was noticed in [16]). The strange feature of the UCNA rate (21) is that it stays τ dependent in the absence of the barrier fluctuations $D_2=0$.

The same system (15)–(17) with rather general potentials $U(x)$ and multiplicative coupling of the Ornstein-Uhlenbeck noise was considered in [2]. There, it was argued that for fixed integral autocorrelation $\int_0^\infty \langle y(t)y(0)\rangle dt$ of the barrier fluctuations, i.e., $D_2(\tau)=\text{const}$, in the limit $\tau\to\infty$ the decay becomes nonexponential and the mean exit time is larger than in the absence of the barrier fluctuations $D_2=0$. However, since in the limit $\tau\to\infty$, the variance $\langle y(t)^2\rangle$ of the barrier fluctuations vanishes for $D_2(\tau)=\text{const}$, it seems obvious to us that this limit is equivalent to switching off the barrier fluctuations, i.e., $D_2=0$, in agreement with (20).

V. SUMMARY AND CONCLUSIONS

We considered the discrete-time dynamics (1) additively disturbed by weak Gaussian white noise (thermal noise) and weak Ornstein-Uhlenbeck noise (barrier fluctuations) of correlation A. For the deterministic part (6) of the dynamics, we chose a piecewise linear map with constant slope B. By means of a transition state theory, we derived the central rate formula (8)–(10) which is exact up to exponentially small corrections in the noise strength ϵ, in agreement with the numerical results in Table I. Then we discussed the escape rate as a function of the correlation A under the usual assumption [4–6,9] that the thermal noise coupling is A independent, $K_1=\text{const}$, whereas the colored-noise coupling may still be A dependent, $K_2=K_2(A)$. For any choice of $K_2(A)$, the qualitative behavior of the rate was shown to be the same for arbitrary values of K_1, in particular for $K_1=0$ describing the escape problem with Ornstein-Uhlenbeck noise alone. Choosing $K_2(A)$ such that the variance $\langle y_n^2\rangle$ of the barrier fluctuations becomes independent of the correlation A, no resonant activation (maximum of the escape rate) was found in contrast to the Doering-Gadoua model [4–6]. With $K_2(A)$ such that the integral autocorrelation $\sum_{n=0}^\infty \langle y_n y_0\rangle$ of the barrier fluctuations becomes A independent, resonant activation occurs provided $\sqrt{5}-2<B<1$. The resonant correlation time is ϵ independent and of the order 1 in our case, unlike the Doering-Gadoua model where it is comparable to the inverse rate [5,6]. A common feature of both models is that resonant activation is not a prefactor effect but is present already in the exponentially leading part of the rate. In the continuous-time limit we found that all the above properties of our model stay true except that resonant activation is absent in the case of fixed integral autocorrelation, at least in the exponentially leading part of the rate.

Similar to the Doering-Gadoua model, the very simple system considered here allows for rigorous results but is rather far from a realistic physical situation of interest. In particular, it is unknown how the intensity of the barrier fluctuations typically behaves as a function of the correlation time, which has been seen to play a crucial role for any kind of "resonant" extremum of the rate. (In the Doering-Gadoua model a correlation-independent variance of the barrier fluctuations is suggestive, but not at all the only possibility.)

A stochastic process describing an overdamped particle in one dimension is known to show a richer behavior in discrete than in continuous time and, as far as weak Gaussian white noise is concerned, is also more difficult to handle [12–14,17,18]. On the other hand, in the case of weak Ornstein-Uhlenbeck noise, the present investigation gives an example for which rigorous results including preexponential contributions are easier to obtain in discrete than in continuous time. We expect that the same stays true also for less simple maps than the one considered here, offering a promising new approach [10] also to the colored-noise debate in continuous time [11]. Work on rigorous results for piecewise linear continuous maps and qualitative results for more general smooth maps is in progress. A variety of rigorous results is not only of interest in order to identify specific and common properties of different models but also in order to check approximative treatments of more general systems, as seen at the end of Sec. IV.

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APPENDIX: CALCULATION OF THE RATE

We first consider the two-dimensional Langevin equation (1) and (2) but with the fully linear map

$$f(x,y)=Bx+y \tag{A1}$$

for all $x\in\mathbb{R}$ instead of (6). The invariant density $W(\mathbf{x})$ is governed by the master equation [19]

$$W(\mathbf{x})=\int_{-\infty}^{\infty}dy_1\int_{-\infty}^{\infty}dy_2 P(\mathbf{x}|\mathbf{y})W(\mathbf{y})\ , \tag{A2}$$

where $\mathbf{x}$ represents a two-dimensional vector with components x_1 and x_2 and similarly for $\mathbf{y}$. In (A2) $P(\mathbf{x}|\mathbf{y})$ is the transition probability to go from $\mathbf{y}$ to $\mathbf{x}$ in one time step and, according to (1) and (2), is given by the Frobenius-Perron equation

$$P(\mathbf{x}|\mathbf{y})=\int_{-\infty}^{\infty}d\xi^{(1)}\int_{-\infty}^{\infty}d\xi^{(2)}P(\xi^{(1)})P(\xi^{(2)})\delta[x_1-f(y_1,y_2)-\sqrt{K_1}\xi^{(1)}]\delta(x_2-Ay_2-\sqrt{K_2}\xi^{(2)})\ . \tag{A3}$$

With (3) and (A1) we infer

$$P(\mathbf{x}|\mathbf{y})=\frac{1}{\pi\epsilon\sqrt{K_1K_2}}\exp\{-[\mathbf{x}-\underline{F}\mathbf{y}]^{+}\underline{K}^{-1}[\mathbf{x}-\underline{F}\mathbf{y}]/\epsilon\}\ , \tag{A4}$$

where the plus sign indicates transposition and where

$$\underline{F}:=\begin{bmatrix}B & 1\\ 0 & A\end{bmatrix},\quad \underline{K}:=\begin{bmatrix}K_1 & 0\\ 0 & K_2\end{bmatrix}\ . \tag{A5}$$

In order to solve the master equation (A2) we use methods introduced in [17,18]: For the invariant density $W(\mathbf{x})$, we make the WKB ansatz

$$W(\mathbf{x})=Z(\mathbf{x})e^{-\phi(\mathbf{x})/\epsilon}\ , \tag{A6}$$

where the ϵ-independent generalized potential $\phi(\mathbf{x})$ describes the exponentially leading part of the invariant density, while the prefactor $Z(\mathbf{x})$ accounts for the nonexponential part and in general still depends on the noise strength ϵ. Combining (A2), (A4), and (A6) yields

$$Z(\mathbf{x})e^{-\phi(\mathbf{x})/\epsilon}=\int_{-\infty}^{\infty}\frac{dy_1}{\sqrt{\pi\epsilon K_1}}\times\int_{-\infty}^{\infty}\frac{dy_2}{\sqrt{\pi\epsilon K_2}}Z(\mathbf{y})e^{-H(\mathbf{x},\mathbf{y})/\epsilon}\ , \tag{A7}$$

where

$$H(\mathbf{x},\mathbf{y}):=\phi(\mathbf{y})+[\mathbf{x}-\underline{F}\mathbf{y}]^{+}\underline{K}^{-1}[\mathbf{x}-\underline{F}\mathbf{y}]\ . \tag{A8}$$

For small ϵ, the rhs of (A7) can be evaluated by means of a saddle-point approximation. Since the prefactor $Z(\mathbf{y})$ is nonexponential in ϵ, comparison of the exponentially leading parts on both sides of (A7) implies

$$\phi(\mathbf{x})=\min_{\mathbf{y}}H(\mathbf{x},\mathbf{y})=:H[\mathbf{x},\mathbf{g}(\mathbf{x})]\ , \tag{A9}$$

where the last equality is an implicit definition of $\mathbf{g}(\mathbf{x})$.

A straightforward but tedious calculation shows that the unique nontrivial solution of (A9) is given by

$$\phi(\mathbf{x})=\mathbf{x}^{+}\underline{\Phi}\mathbf{x},\quad \mathbf{g}(\mathbf{x})=\underline{F}^{-1}(\underline{1}-\underline{K}\underline{\Phi})\mathbf{x}\ , \tag{A10}$$

where $\underline{1}$ is the unit matrix and

$$\underline{\Phi}=\frac{\begin{bmatrix}(1-AB)^2(1-B^2) & -A(1-AB)(1-B^2)\\ -A(1-AB)(1-B^2) & 1-A^2B^2+\dfrac{K_1}{K_2}(1-A^2)(1-AB)^2\end{bmatrix}}{(1-AB)^2K_1+K_2} \tag{A11}$$

A more natural proof that (A10) and (A11) solve (A9) and, in particular, the derivation of this solution is conveniently carried out within the framework outlined in [18] (similar to the continuous-time case [20], one has to solve an inhomogeneous linear equation for $\underline{\Phi}^{-1}$). Due to (5) and (7), the matrix $\underline{\Phi}$ in (A11) is positive definite. The matrix $\underline{F}^{-1}(\underline{1}-\underline{K}\underline{\Phi})$ in (A10) stays finite even in the limits $A\to 0$ and $B\to 0$ where $\underline{F}^{-1}$ in (A5) diverges. With (A10) and (A11) it follows that $H(\mathbf{x},\mathbf{y})$ in (A8) is of the form

$$H(\mathbf{x},\mathbf{y})=\phi(\mathbf{x})+[\mathbf{y}-\mathbf{g}(\mathbf{x})]^{+}\underline{H}[\mathbf{y}-\mathbf{g}(\mathbf{x})]\ , \tag{A12}$$

$$\underline{H}:=\underline{\Phi}+\underline{F}^{+}\underline{K}^{-1}\underline{F}\ . \tag{A13}$$

Now the saddle-point approximation on the rhs of (A7) can be carried out, implying

$$Z(\mathbf{x})=Z[\mathbf{g}(\mathbf{x})][K_1K_2\mathrm{Det}(\underline{H})]^{-1/2} .$$

A simple calculation yields $\mathrm{Det}(\underline{H})=(K_1K_2)^{-1}$. Hence we arrive at

$$Z(\mathbf{x})=\mathrm{const}=N , \tag{A14}$$

where N is a normalization constant. By insertion into the master equation (A7), one finds that our solution (A10) and (A14) is actually not only valid within the saddle-point approximation, i.e., for small ϵ, but also is strictly correct for arbitrary ϵ. In other words, for the globally linear dynamics (A1) the invariant density is given by a Gaussian distribution about the point attractor at the origin, as was to be expected. In passing, we note that by integration of the invariant density (A6), (A10), and (A14) over x_1, the following reduced density $W(y)$ for the $x_2=y$ dynamics (2) is found:

$$W(y)=\left[\frac{1-A^2}{\pi\epsilon K_2}\right]^{1/2}\exp\left\{-\frac{1-A^2}{\epsilon K_2}y^2\right\} . \tag{A15}$$

From the saddle-point approximation on the rhs of (27) about $\mathbf{y}=\mathbf{g}(\mathbf{x})$, we can infer that for small noise strengths ϵ, almost all particles which visit the point $\mathbf{x}$ at a certain time step were close to the point $\mathbf{y}=\mathbf{g}(\mathbf{x})$ at the preceding time step. In other words, they arrived along the most probable path $\mathbf{g}(\mathbf{x}),\mathbf{g}(\mathbf{g}(\mathbf{x})),\ldots$ at the end point $\mathbf{x}$. From (A8)–(A10) it follows that

$$\phi(\mathbf{x})=\phi[\mathbf{g}(\mathbf{x})]+\mathbf{x}^{+}\underline{\Phi}\underline{K}\underline{\Phi}\mathbf{x} . \tag{A16}$$

Since $\underline{\Phi}\underline{K}\underline{\Phi}$ is positive definite, the generalized potential obeys $\phi[\mathbf{g}(\mathbf{x})]<\phi(\mathbf{x})$ except for the trivial case $\mathbf{x}=\mathbf{g}(\mathbf{x})=\mathbf{0}$. Consequently, for any $\mathbf{x}\neq\mathbf{0}$, the generalized potential along the most probable path $\mathbf{g}(\mathbf{x}),\mathbf{g}[\mathbf{g}(\mathbf{x})],\ldots$ is strictly smaller than at the end point $\mathbf{x}$.

The generalized potential $\phi(\mathbf{x})$ in (A10) can be rewritten as

$$\phi(\mathbf{x})=\Delta\phi x_1^2+\Phi_{22}\left[x_2-\frac{\Phi_{12}}{\Phi_{22}}x_1\right]^2 , \tag{A17}$$

where Φ_{ij} are the components of Φ and

$$\Delta\phi:=\frac{\mathrm{Det}(\underline{\Phi})}{\Phi_{22}}=\frac{(1-A^2)(1-AB)(1-B^2)}{(1-A^2)(1-AB)K_1+(1+AB)K_2} . \tag{A18}$$

Note that the positive definiteness of $\underline{\Phi}$ guarantees $\Phi_{22}>0$. Restricting ourselves to the half-plain

$$G:=[1,\infty]\times\mathbb{R} , \tag{A19}$$

the generalized potential (A17) has a global minimum at the point

$$\mathbf{x}_s:=(1,\Phi_{12}/\Phi_{22}) , \tag{A20}$$

i.e., $\phi(\mathbf{x})>\phi(\mathbf{x}_s)$ for all $\mathbf{x}\in G\setminus\mathbf{x}_s$. Thus the most probable path to the end point $\mathbf{x}_s$ is entirely contained in the complement $\overline{G}:=\mathbb{R}^2\setminus G$ of G. The same property applies for all end points $\mathbf{x}$ in a small neighborhood of $\mathbf{x}_s$ because of continuity reasons and obviously for all end points $\mathbf{x}$ with $\phi(\mathbf{x})<\phi(\mathbf{x}_s)$, in particular for $\mathbf{x}$ within a small neighborhood of the origin.

Next we return to the escape problem for the piecewise linear model (6) of Sec. II. We first consider the situation that all particles which enter the domain G in (A19) are removed at the next time step. Thus the dynamics of the nonabsorbed particles within $\overline{G}$ is still described by the map (A1). For small noise strengths ϵ the system approaches a pseudoinvariant state described by a pseudoinvariant density $\tilde{W}(\mathbf{x})$ [13]. The escape rate k from $\overline{G}$ into G is given by the number of particles within G (which are removed at the next time step) divided by the population within $\overline{G}$:

$$k=\frac{\int_G\tilde{W}(\mathbf{x})d\mathbf{x}}{\int_{\overline{G}}\tilde{W}(\mathbf{x})d\mathbf{x}} . \tag{A21}$$

For small noise strengths ϵ and neglecting normalization constants, the probability $\tilde{W}(\mathbf{x})$ to be at a certain point $\mathbf{x}$ is equal to the invariant density $W(\mathbf{x})$ of the fully linear system (A1) provided the most probable path $\mathbf{g}(\mathbf{x})$, $\mathbf{g}[\mathbf{g}(\mathbf{x})],\ldots$ is entirely contained in the domain $\overline{G}$ since in this domain the piecewise and the fully linear dynamics agree. As seen above, this is in particular true for end points $\mathbf{x}$ within small neighborhoods of the origin and of the point $\mathbf{x}_s$ in (A20). In the opposite case where the most probable path of the fully linear problem leaves $\overline{G}$, one has $\tilde{W}(\mathbf{x})<W(\mathbf{x})$ since this path does not contribute to $\tilde{W}(\mathbf{x})$. Taking into account that within $\overline{G}$ and G the invariant density $W(\mathbf{x})$ has very pronounced absolute maxima at $\mathbf{x}=0$ and $\mathbf{x}=\mathbf{x}_s$, respectively, it follows that $\tilde{W}(\mathbf{x})$ in the rate formula (A21) can be replaced by $W(\mathbf{x})$ up to exponentially small errors in ϵ, i.e., up to a factor $1+O(e^{-c/\epsilon})$ with an ϵ-independent $c>0$. By means of the above results for $W(\mathbf{x})$, a straightforward calculation yields the final rate formula (8)–(10).

Clearly, the dominant contribution to the rate (A21) stems from particles which escape into a close neighborhood of $\mathbf{x}_s$ within G. If they were not removed at the next time step, the overwhelming majority of them would closely follow a deterministic path during the subsequent time steps. From the piecewise linear dynamics (6), it then easily follows that they arrive at a small neighborhood of the point attractor at $x=x_1=2$, $y=x_2=0$ within a few time steps. In particular, recrossings into the region $\overline{G}$ can be neglected once a particle has left $\overline{G}$. Thus, the rate (8) is correct up to exponentially small corrections in ϵ also with respect to escapes from $\overline{G}$ into a small neighborhood of the point attractor at $\mathbf{x}=(2,0)$.

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